

**Definition.** The general **first order linear ODE** in the unknown function  $x = x(t)$  has the form:

$$A(t) \frac{dx}{dt} + B(t)x(t) = C(t). \quad (1)$$

As long as  $A(t) \neq 0$  we can simplify the equation by dividing by  $A(t)$ .

$$\frac{dx}{dt} + p(t)x(t) = q(t) \quad (2)$$

We'll call (2) the **standard form** for a first order linear ODE.

## 1. Terminology and Notation

The functions  $A(t)$ ,  $B(t)$  in (1), and  $p(t)$  in (2), are called the **coefficients** of the ODE. If  $A$  and  $B$  (or  $p$ ) are constants (i.e. do not depend on the variable  $t$ ) we say the equation is a **constant coefficient** DE.

We use the familiar notations  $y'$  or  $\dot{y}$  for the derivative of  $y$ . With some exceptions, we'll use  $\dot{y} = \frac{dy}{dt}$  to mean the derivative with respect to time and  $y'$  for derivatives with respect to some other variable, e.g.  $y' = \frac{dy}{dx}$ . If there is any danger of confusion we'll revert to the unambiguous Liebnitz notation:  $\frac{dy}{dt}$ ,  $\frac{dy}{dx}$ , etc.

## 2. Homogeneous/Inhomogeneous

If  $C(t) = 0$  in (1) the resulting equation:

$$A(t)\dot{x} + B(t)x = 0$$

is called **homogeneous**<sup>1</sup>. Likewise, in standard form,  $\dot{x} + p(t)x = 0$  is homogeneous. Otherwise the equation is **inhomogeneous**.

**!► Example 5.1:** Consider the differential equation

$$x \frac{dy}{dx} + 4y - x^3 = 0 \quad .$$

Solving for the derivative, we get

$$\frac{dy}{dx} = \frac{x^3 - 4y}{x} = x^2 - \frac{4}{x}y \quad ,$$

which is

$$\frac{dy}{dx} = f(x) - p(x)y$$

with

$$p(x) = \frac{4}{x} \quad \text{and} \quad f(x) = x^2 \quad .$$

So this first-order differential equation is linear. Adding  $\frac{4}{x} \cdot y$  to both sides, we then get the equation in standard form,

$$\frac{dy}{dx} + \frac{4}{x}y = x^2 \quad ,$$

On the other hand

$$\frac{dy}{dx} + \frac{4}{x}y^2 = x^2$$

is not linear because of the  $y^2$ .

In testing whether a given first-order differential equation is linear, it does not matter whether you attempt to rewrite the equation as

$$\frac{dy}{dx} = f(x) - p(x)y$$

or as

$$\frac{dy}{dx} + p(x)y = f(x) \quad .$$

If you can put it into either form, the equation is linear. You may prefer the first, simply because it is a natural form to look for after solving the equation for the derivative. However, because the second form (the standard form) is more suited for the methods normally used for solving these equations, more experienced workers typically prefer that form.

If a differential equation cannot be written in the form, [\(1\)](#) then it is called a **non-linear** differential equation.

$\sin(y) \frac{d^2y}{dx^2} = (1-y) \frac{dy}{dx} + y^2 e^{-5y}$  be an example of non linear ODE.